# Mathematics for Data Analysis
## – Combinatorics –

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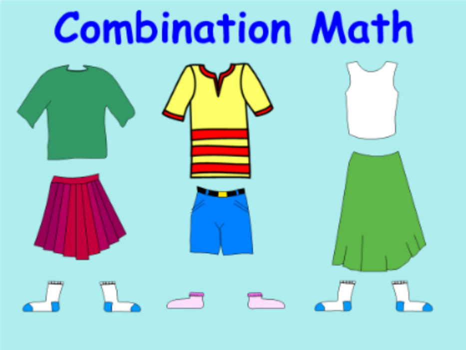
## Why Combinatorics?

When we perform an experiment whose outcome cannot be forecasted (like a coin toss or rolling dices), if the number of possible outcomes is finite and there is no reason to assume that the outcomes are not equally possible (for instance, using fair coins or dices), then combinatorics gives a way of computing the probability of a specified outcome.

## Combinatorics

Combinatorics studies the ways of choosing some objects out of a finite collection and/or number of ways of their arrangement.



**Problem: athletics competition**

How many ways can three people take turns on the podium in a race with 10 participants?

**Solution**

Out of ten participants, one will finish in first place. How can the second and third place be chosen? The runner-up can be chosen from the remaining nine people; the third among the remaining eight. Then, we have

$$10 \cdot 9 \cdot 8 = 720$$

possible podiums.

**Example**

Suppose there are 5 members in a club, denoted with A, B, C, D, and E, and one of them is to be chosen as the coordinator. Clearly any one out of them can be chosen so there are 5 ways.
Now suppose two members are to be chosen for the position of coordinator and co-coordinator. We can choose A as coordinator and one out of the remaining 4 as co-coordinator, and similarly with B, C, D and E. So there will be total 20 possible ways.

**Remark**

Note that choosing A then B and choosing B then A are considered different, i.e., the way of arrangement is important. If two coordinators are to be chosen, so here choosing A then B and choosing B then A will be same. Number of different ways here will be 10.

# Definitions

Consider a finite set of $n$ ($n > 0$) elements that we denote as $a_1, a_2, a_3, \ldots, a_n$, or simply with their index $(1, 2, 3, \ldots, n)$.
Combinatorics allows us to determine the number of ways these elements can be redistributed into groups of $k \leq n$ elements.

The groups that we may consider are: permutations and combinations. Both can be simple or with repetition.

## Permutations (simple or without repetition)

Permutations occur when we want to count the ways we can choose $k$ elements out of $n$ elements ($k \leq n$), and we consider two permutations are different if they do not contain the same elements, or have the same elements arranged differently.

The number of the permutations without repetition, *i.e.*, the number of arrangements of $k$ items from $n$ elements, is

$$P_{n,k} = n(n-1)(n-2)\cdot\dots\cdot(n-k+1).$$

In fact, in each permutation, we can choose the first element in $n$ different ways, the second element in $(n-1)$ different ways, ..., the $k$–th element in $(n-(k-1))$ ways.

## Full permutation

If $k = n$, we have the full permutation of $n$ different elements, that is

$$P_{n,n} = n(n-1)(n-2)\cdots 2\cdot 1 = n! \qquad \text{(factorial } n\text{)}.$$

## Factorial and properties

The factorial of a non-negative integer $n$, denoted by $n!$, is the product of all positive integers less than or equal to $n$. The factorial of $n$ also equals the product of $n$ with the next smaller factorial:

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1 = n \cdot (n-1)!$$

Furthermore, it is

$$0! = 1,$$

according to the convention for an empty product.

## Examples

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$
$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$
$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3628800$$

## Exercises

Compute

$$\frac{20! + 21!}{19!}$$

Verify that:

$$1)\quad n^2(n-1)! + n! = (n+1)!, \quad \forall n \in \mathbb{N},\ n > 0;$$

$$2)\quad n \cdot n! - (n+1)! = -n! \quad \forall n \in \mathbb{N};$$

$$3)\quad n! + (n+1)! + (n-1)! = (n+1)^2(n-1)! \quad \forall n \in \mathbb{N},\ n > 0;$$

$$4)\quad (n+1)! - n! = \frac{(n!)^2}{(n-1)!} \quad \forall n \in \mathbb{N},\ n > 0.$$

## Problem

How many ways we can choose 2 elements out of the set $\{1, 2, 3\}$?

## Solution

$$P_{3,2} = 3 \cdot 2 = 6,$$

that are

$$(1, 2), \qquad (1, 3), \qquad (2, 1), \qquad (2, 3), \qquad (3, 1), \qquad (3, 2).$$

### Problem

How many ways we can choose 3 elements out of the set $\{1, 2, 3\}$?

### Solution

$$P_{3,3} = 3! = 3 \cdot 2 \cdot 1 = 6,$$

that are

$$(1, 2, 3), \quad (1, 3, 2), \quad (2, 1, 3), \quad (2, 3, 1), \quad (3, 1, 2), \quad (3, 2, 1).$$

### Problem

How many ways can 6 people sit in 10 numbered chairs?

### Solution

The first person can choose 1 out of the 10 chairs, the second one 1 out of the remaining 9 chairs, ...and so on. Therefore, we have to compute the permutations of 6 elements out of 10:

$$P_{10,6} = 10 \cdot (10-1) \cdot (10-2) \cdot (10-3) \cdot (10-4) \cdot (10-5) = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151200.$$

If we have 6 people and 6 numbered seats, then we need to compute

$$P_{6,6} = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

### Problem

How many anagrams, even without meaning, can be formed with the word LIBER?

### Solution

We can choose the first letter in five different ways, the second in four ways, the third in three ways, the fourth in two ways, and the last in only one way. Therefore, we obtain

$$P_{5,5} = 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \quad \text{anagrams.}$$

We have considered all possible groups of 5 distinct elements out of 5 distinct elements.

## Permutations with repetition of $n$ elements

If in permutations of $k$ elements out of $n$ we admit that the elements may occur more than once, the formula giving their number is

$$P'_{n,k} = n^k.$$

In fact, the first element of each permutation can be chosen in $n$ possible ways, and so the second one, the third one, ..., the $k$-th one, *i.e.*,

$$\underbrace{n \cdot n \cdot \ldots \cdot n}_{k \text{ times}} = n^k$$

possible groupings.

**Problem**

Consider a set of 4 colors: white, yellow, red and green. How many groups of 3 colors not necessarily different can we construct?

**Solution**

Since a color may appear more than once in each group, we have the permutations with repetition of 3 elements out of 4 objects, that is

$$P'_{4,3} = 4 \cdot 4 \cdot 4 = 4^3 = 64.$$

## Problem

How many numbers having three odd digits may we construct?

## Solution

There are 5 odd digits and with them we may construct

$$P'_{5,3} = 5^3 = 125 \qquad \text{numbers.}$$

### Problem

How many numbers having three digits and divisible by 5 can we construct?

### Solution

The first digit (from left to right) can be chosen among 9 non vanishing digits, and the second digit can be chosen among all 10 digits. The last digit (less significant) may be 0 or 5.

Then, the solution is

$$9 \cdot 10 \cdot 2 = 180.$$

## Problem

How many anagrams (also nonsense) can we construct with the word COMPUTER?

## Solution

We have to compute all the permutations of the 8 letters in the word COMPUTER, so that

$$P_{8,8} = 8! = 40320.$$

## Permutations with repetition of $n$ elements

If among the $n$ elements we have $m \leq n$ equal elements, we have to remove the equal permutations from the $n!$ possible permutations. The number of equal permutations are $m!$; thus, we need to divide the total number of permutations by $m!$, that is

$$P_{n,n}^{(m)} = \frac{n!}{m!}.$$

## Example

How many anagrams (also nonsense) can be constructed from the word AREA?

## Solution

We have 4 letters ($n = 4$), but two letters ($m = 2$) are repeated, whence the anagrams are AREA, AAER, AARE, AEAR, AERA, ARAE, EAAR, EARA, ERAA, RAAE, RAEA, REAA, and their number is

$$P_{4,4}^{(2)} = \frac{4!}{2!} = \frac{24}{2} = 12.$$

## Generalization: full permutations of a finite number of repeated elements

Let $n$ elements be divided into $k$ groups. The elements in the same group are the same as each other, and the elements in different groups are different. Let the number of elements in $k$ groups be $n_1$, $n_2, \ldots,$ $n_k$ ($n_1$ objects are of type 1, $n_2$ objects are of type 2, $\ldots$, $n_k$ objects are of type $k$), with $n_1 + n_2 + \ldots + n_k = n$.
Then, the number of ways of arrangement of these $n$ objects, *i.e.*, the number of a full permutation of a finite number of repeated elements is given by

$$P_{n,n}^{(n_1,n_2,\ldots,n_k)} = \frac{n!}{n_1!\, n_2! \cdots n_k!}.$$

## Example

How many anagrams (also nonsense) of the word MATHEMATICS can be done?

## Solution

The word MATHEMATICS has 11 letters not all distinct. There are two A's, two M's and two T's. Here we have $n_1 = n_2 = n_3 = 2$, and the number of distinct anagrams is

$$\frac{11!}{2! \cdot 2! \cdot 2!} = 4989600.$$

## Problem

Among all the numbers of 10 different digits, how many are the ones whose first 5 digits are odd?

## Solution

Since the digits are all different and the first 5 are odd, the last 5 digits must be even. All possible groups made by 5 odd digits are $P_{5,5} = 5!$ and, then, also all possible groups made by 5 even digits are $P_{5,5} = 5!$. So, the answer is

$$5! \cdot 5! = 14400.$$

**Problem**

There are 12 numbered seats in a bus. In how many ways 5 different people can occupy them?

**Solution**

$$P_{12,5} = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 95040.$$

## Problem

Among all the 3-digit numbers, all odd and different from each other, how many are there multiples of 5?

## Solution

The 3-digit numbers with odd and different digits are $5 \cdot 4 \cdot 3 = 60$. But, multiples of 5 are those that have the last digit equal to 5. Therefore, the first digit can be chosen in 4 different ways and the second in 3 different ways (the digit 5 is constrained to be the last digit). The answer is

$$P_{4,2} = 4 \cdot 3 = 12.$$

### Problem

How many four-digit numbers can be formed with the numbers 1,5,6,8,9?

### Solution

The answer is

$$P_{5,4} = 5 \cdot 4 \cdot 3 \cdot 2 = 120.$$

## Problem

Determine all groups of 3 distinct elements out of the set $\{1,2,3,4\}$ which are distinguished by at least one of the elements but not for the arrangement!

## Solution

The permutations of 3 elements out of 4 are

$$P_{4,3} = 4 \cdot 3 \cdot 2 = 24,$$

*i.e.*,

$$\begin{array}{cccccc}
(1,2,3), & (1,2,4), & (1,3,2), & (1,3,4), & (1,4,2), & (1,4,3), \\
(2,1,3), & (2,1,4), & (2,3,1), & (2,3,4), & (2,4,1), & (2,4,3), \\
(3,1,2), & (3,1,4), & (3,2,1), & (3,2,4), & (3,4,1), & (3,4,2), \\
(4,1,2), & (4,1,3), & (4,2,1), & (4,2,3), & (4,3,1), & (4,3,2).
\end{array}$$

## . . . Solution

But we look for groups that are not distinguished by order.
The pairs of each row

$$
\begin{array}{cccccc}
(1,2,3), & (1,3,2), & (2,1,3), & (2,3,1), & (3,1,2), & (3,2,1), \\
(1,2,4), & (1,4,2), & (2,1,4), & (2,4,1), & (4,1,2), & (4,2,1), \\
(1,3,4), & (1,4,3), & (3,1,4), & (3,4,1), & (4,1,3), & (4,3,1), \\
(2,3,4), & (2,4,3), & (3,2,4), & (3,4,2), & (4,2,3), & (4,3,2),
\end{array}
$$

represent the same group and are considered only once.
The combinations of 3 elements out of 4 are

$$C_{4,3} = \frac{P_{4,3}}{P_{3,3}} = \frac{4 \cdot 3 \cdot 2}{3!} = \frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1} = 4.$$

## Combinations (simple or without repetition)

Combinations occur when we want to count the ways we can choose $k$ elements out of $n$ elements ($k \leq n$).
Two combinations are different if they do not contain the same elements; two combinations with the same elements but arranged differently do not differ.
The number of combinations without repetition of $k$ items out of $n$ different elements is

$$C_{n,k} = \frac{P_{n,k}}{P_{k,k}} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

In fact, the number of combinations can be obtained by the ratio of the number of permutations of $k$ elements chosen out of $n$ elements with the number of the permutations of $k$ elements.

If $k = n$, of course we have

$$C_{n,n} = \binom{n}{n} = 1.$$

## Binomial coefficient

The quantity

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

is called binomial coefficient, since it occurs in the formula of the $n$–th power of a binomial:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

and it is related to Pascal's triangle (in Italy known as Tartaglia's triangle):

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
⋮

## Pascal's triangle

For $n = 0, 1, 2, \ldots$ and writing on subsequent lines the corresponding binomial coefficients $\binom{n}{k}$, $k \leq n$, we obtain the Pascal's triangle:

$$
\begin{array}{ll}
n = 0 & \binom{0}{0} \\
n = 1 & \binom{1}{0} \quad \binom{1}{1} \\
n = 2 & \binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
n = 3 & \binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
n = 4 & \binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
n = 5 & \binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
n = 6 & \binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
 & \vdots
\end{array}
$$

## Example: binomial coefficient

$$(a+b)^4 = \sum_{k=0}^{n} \binom{4}{k} a^{4-k} b^k = \binom{4}{0} a^4 + \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 + \binom{4}{3} ab^3 + \binom{4}{4} b^4 =$$
$$= a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4.$$

## Remark

The number of the permutations without repetition of $k$ items from $n$ elements can be also written as

$$P_{n,k} = \frac{n!}{(n-k)!}$$

In fact:

$$P_{n,k} = n(n-1)(n-2)\cdot \dots \cdot (n-k+1) =$$
$$= \frac{n(n-1)(n-2)\cdot \dots \cdot (n-k+1)\cdot (n-k)\cdot (n-k-1)\cdot \dots \cdot 2\cdot 1}{(n-k)\cdot (n-k-1)\cdot \dots \cdot 2\cdot 1} = \frac{n!}{(n-k)!}$$

## Binomial coefficient: properties

$$\binom{n}{1} = \binom{n}{n-1} = n, \qquad \binom{n}{0} = \binom{n}{n} = 1,$$

$$\binom{n}{k} = \binom{n}{n-k}, \qquad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}, \qquad \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}.$$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} =$$

$$= n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n-k+1}{k}.$$

## Problem

In the bingo game (numbers from 1 to 90), how many combinations of 6 numbers chosen out of 90 can we count?

## Solution

The answer is

$$C_{90,6} = \binom{90}{6} = \frac{90!}{6! \cdot 84!} = 622614630.$$

## Problem

At the end of a dinner 7 friends greet each other by exchanging handshakes. How many handshakes do they exchange?

## Solution

In such a case, in counting handshakes the order of two friends greeting each other does not matter. Thus, the number of handshakes is

$$C_{7,2} = \binom{7}{2} = \frac{7!}{2! \cdot 5!} = \frac{7 \cdot 6 \cdot 5!}{2! \cdot 5!} = \frac{7 \cdot 6}{2} = 21.$$

## Problem

How many ambi, terni, quaternities can be formed with the ninety bingo numbers?

## Solution

$$C_{90,2} = \binom{90}{2} = \frac{90!}{2! \cdot 88!} = 4005,$$

$$C_{90,3} = \binom{90}{3} = \frac{90!}{3! \cdot 87!} = 117480,$$

$$C_{90,4} = \binom{90}{4} = \frac{90!}{4! \cdot 86!} = 2555190.$$

## Problem

10 cards are shuffled and 5 are dealt to player A and 5 to player B. How many different ways the distribution of cards can take place?

## Solution

The combinations $C_{10,5}$ of 5 elements out of 10 elements are the cards received by player A; the remaining cards are assigned to player B. Then, we have

$$C_{10,5} \cdot \underbrace{C_{5,5}}_{=1} = \binom{10}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 5!} = 252$$

different ways to distribute the cards.

## Problem

12 cards are shuffled and 3 are dealt to player A, 3 to player B, 3 to player C and 3 to player D. How many different ways the distribution of cards can take place?

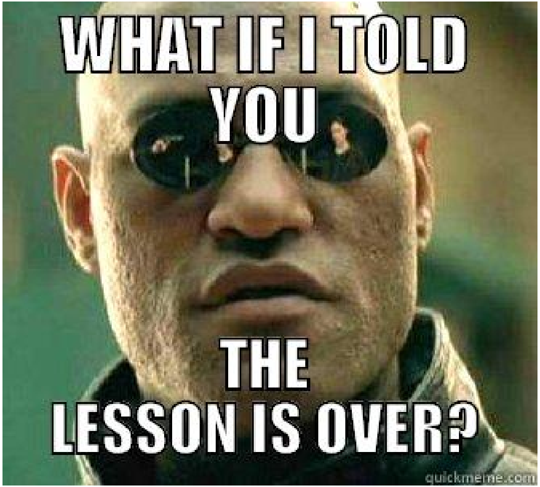
## Solution

$C_{12,3}$ are the cards received by player A, $C_{9,3}$ are the cards received by player B, $C_{6,3}$ are the cards received by player C. Player D receives the remaining cards. Then, we have

$$C_{12,3} \cdot C_{9,3} \cdot C_{6,3} \cdot \underbrace{C_{3,3}}_{=1} = \binom{12}{3} \cdot \binom{9}{3} \cdot \binom{6}{3} = 369600$$

different ways.

# Tired?



## Combinations with repetition

Also combinations may have repeated elements!
Given $n$ distinct elements, each group differs from the others for at least one element or for the number of repetitions that occurs with an element.

In other words, a combination with repetition is a sample of $k$ elements from a set of $n$ elements allowing for duplicates but disregarding different orderings (*e.g.* $\{2,1,2\} = \{1,2,2\}$).

We have

$$C'_{n,k} = \frac{n \cdot (n+1) \cdot (n+2) \cdot \ldots \cdot (n+k-1)}{k!} = \frac{(n+k-1)!}{k! \cdot (n-1)!} = \binom{n+k-1}{k}.$$

## Problem

By using the digits 1 and 2, how many groups of 3 digits is it possible to create so that each group differs from the other for an element or for the number of repetitions of an element?

## Solution

$$C'_{2,3} = \frac{(2+3-1)!}{3!1!} = \frac{4!}{3!} = 4.$$

The 3-digit groups are:

$$(1,1,1), \quad (1,1,2), \quad (1,2,2), \quad (2,2,2).$$

### Problem: subsets

How many subsets made by 5 elements can be extracted from a set with 20 elements?

### Solution

In sets, the order of the elements is not important. Thus, the number of the required subsets are the combinations (without repetition) of 5 elements chosen out of 20:

$$C_{20,5} = \binom{20}{5} = 15504.$$

### Problem

Assigned two droppers, the first containing 5 drops of white color and the second 5 drops of black color. By mixing together 5 drops chosen between the two colors, how many different colors can be formed?

### Solution

$$C'_{2,5} = \frac{(2+5-1)!}{5!(2-1)!} = \frac{6!}{5!} = 6.$$

Only 6 colors can be formed: white, black and 4 shades of gray.

### Problem

5 red, 2 white and 3 blue balls must be placed in a row. If all balls of the same color are indistinguishable, how many arrangements are possible?

### Solution

We have 10 balls that can be arranged in 10! different ways. Since balls of the same color are indistinguishable, we have to divide by $5! \cdot 2! \cdot 3!$. Then, we have

$$\frac{10!}{5! \cdot 2! \cdot 3!} = 2520$$

arrangements.

## Problem

How many different ways can 5 boys and 6 girls be arranged in a row of chairs, with the condition that the boys are all close together as well as the girls?

## Solution

The number of ways of accommodation for the 5 boys is given by $P_{5,5}$, wherease the number of ways of accommodation for the 6 girls is given by $P_{6,6}$. Furthermore, since you can arrange the boys first and then the girls, or vice versa, the number of arrangements for the 5 boys and 6 girls is:

$$2 \cdot P_{5,5} \cdot P_{6,6} = 2 \cdot 5! \cdot 6! = 2 \cdot 120 \cdot 720 = 172800.$$

## Problem

An urn contains 20 numbered balls. In how many ways can we extract 3 balls, supposing that after each extraction the ball is returned to the urn, and without taking into account the arrangement of the extracted balls?

## Solution

We have $n = 20$ distinct elements. Each ball can be extracted at most 3 times, then we need to compute the combinations with repetitions of 3 elements chosen out of 20:

$$C'_{20,3} = \binom{20+3-1}{3} = \binom{22}{3} = \frac{22!}{3!\cdot 19!} = \frac{22\cdot 21\cdot 20\cdot 19!}{3!\cdot 19!} = \frac{20\cdot 21\cdot 22}{6} = 1540.$$

## Power set

In mathematics, the power set of a set $S$, denoted by $\mathcal{P}(S)$, is the set of all subsets of $S$, including the empty set $\emptyset$ and $S$ itself.

## Problem

How many elements in the power set of a set $S$ with $n$ elements?

**Solution.**

It is:

$$
\begin{aligned}
|\mathcal{P}(S)| &= 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} + 1 = \\
&= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} + \binom{n}{n} = \\
&= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^{k} = (1+1)^n = 2^n.
\end{aligned}
$$

### Example

Given the set $S = \{1, 2, 3, 4\}$, it is

$$|\mathcal{P}(S)| = 2^4 = 16.$$

In fact, the power set of $S$ is given by

$$\mathcal{P}(S) = \{\{\emptyset\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\},$$
$$\{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}.$$

### Problem

Using the digits 1,2,3, how many 6-digit numbers can be formed?

### Solution

$$P'_{3,6} = 3^6 = 729.$$

### Problem

A factory produces one-liter cans of paint by mixing 10 one-deciliter scoops each of red, yellow, and blue. How many packages of paints are produced?

### Solution

Each type of paint produced is a combination with repetition of $n = 3$ objects (the basic colors) chosen out of $k = 10$ elements (the number of independent deciliters to be mixed). With this procedure,

$$C'_{n,k} = \binom{n+k-1}{k} = \binom{3+10-1}{10} =$$
$$= \binom{12}{10} = \frac{12!}{2!10!} = \frac{12 \cdot 11}{2} = 66$$

packages of paints are produced.

## Problem

How many colors are obtained by mixing 100 independent centilitres?

## Solution

$$C'_{n,k} = \binom{n+k-1}{k} = \binom{3+100-1}{100} = $$
$$= \binom{102}{100} = \frac{102!}{2!100!} = 5151.$$

## Problem

Consider an extraction in a bingo game. How many possible five-numbers sets containing the numbers 1 and 90 are there?

## Solution

They are the combinations of 3 elements chosen out of 3:

$$C_{88,3} = \frac{88!}{3!85!} = 109736.$$

**Problem**

Four tennis players want to play a doubles. How many teams can be formed?

**Solution**

The answer is

$$C_{4,2} = \frac{4!}{2!2!} = \frac{4 \cdot 3 \cdot 2}{2 \cdot 2} = 6.$$

## Problem

A shopkeeper wants to display 4 pairs of shoes chosen from 10 different models in a shop window. In how many ways (the order does not count) can shoes be displayed in the showcase?

## Solution

$$C_{10,4} = \frac{10!}{4!6!} = 210.$$

## Problem

How many ways can you arrange the cards in a 40-card deck?

## Solution

$$P_{40,40} = 40!$$

## Problem

How many games of chess can six players play?

## Solution

$$C_{6,2} = \frac{6!}{2!4!} = \frac{6 \cdot 5}{2} = 15.$$

## Problem

The Italian alphabet contains 16 consonants and 5 vowels. How many strings of 6 letters can be formed with all different letters so that they contain the letters "a" and "b"?

## Solution

The answer is

$$2 \cdot C_{6,2} \cdot P_{19,4} = 2 \cdot \binom{6}{2} \cdot 19 \cdot 18 \cdot 17 \cdot 16 = 2 \cdot 15 \cdot 19 \cdot 18 \cdot 17 \cdot 16 = 2790720.$$

## Problem

Suppose a restaurant menu consists of 5 appetizers, 6 first courses, 6 second courses and 4 desserts: how many complete meals (of 4 dishes) can we order?

## Solution

The answer is

$$5 \cdot 6 \cdot 6 \cdot 4 = 720.$$